Research Statement

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1. Introduction

My principal area of research interest is that of *inverse problems* in partial differential equations. Briefly, an inverse problem may be described as a problem of *non-invasive* or *non-accessible determination*, referring to determination of information about the interior of a body without having access to the interior, or even the body itself. This requires that one sends waves through the interior in the form of light waves, x-rays, or ultrasound, or that one makes electrostatic boundary measurements. One then analyzes the results in order to determine the properties of the interior. The *forward problem* is the prediction, with prior knowledge of the interior of the body, of the values of the measurements observed on the boundary. The inverse problem is to determine the interior from knowledge only of these boundary measurements. This is typically much more difficult, and even proving that the observed results can come from only one interior can be quite difficult. This is the question of uniqueness.

Examples of well-known inverse problems include x-ray tomography, ultrasound imaging, MRI imaging, and earth sub-structure exploration. The potential practical applications make inverse problems of active interest to both the academic and applied worlds. Furthermore, the natural accessibility and the importance of their practical applications make inverse problems particularly easy to explain to undergraduates and to the public in general. Thus my research interests are able to be incorporated into all facets of my academic activities.

My doctoral research was the study of an inverse problem arising in electrodynamics. When an electromagnetic body is immersed into a time-dependent electric field, a time-dependent magnetic field is induced. The relationship between these fields is described by Maxwell's equations which depend on the electromagnetic parameters of the body, for example, the conductivity. Typically, three such parameters are considered; a fourth physically significant parameter, the *chirality* of the body, is often neglected. Presence of chirality results in a rotation of the electromagnetic fields, and requires an alteration in the form of Maxwell's equations. I have proven that for such a body, the chirality and other parameters are uniquely determined by information obtainable at the boundary. In other words, two different chiral electromagnetic bodies cannot give rise to the same boundary information. The underlying techniques involved in the proof of this result are those of microlocal analysis. A more precise statement of the theorem and further related results are included below.

Professor Adrian Nachman and myself are currently involved in an exciting research project on which we have been working for the last year or more. Dr. Nachman has been collaborating for several years with Professor Waag of the electrical and computing engineering department on the problem of reconstructing images of human tissue from experimental data recorded by Professor Waag's state of the art ultrasound instrument. I have joined this team and we are seeking to develop a numerically stable algorithm to produce such images. The implications of this research are far-reaching and we expect new and ground-breaking results in the near future. The research encompasses questions in analysis, numerical implementation, and practical application.

I am currently also working on an inverse problem in Riemannian geometry. Briefly, given a Riemannian metric on a manifold, one has an elliptic operator termed the Laplace-Beltrami operator which is a generalization of the Euclidean Laplacian. This operator can be extended to act on p-forms. A significant open question in inverse problems is whether a metric can be determined from boundary information for the Laplace-Beltrami operator on functions. I have been looking at the problem of determining the metric from boundary information for this operator acting on p-forms, restricting myself to the case of metrics conformal to the Euclidean.

Other research activities include working with a graduate student from the geology department on determining the strain tensor of deformed rock samples from field measurements. I have also worked with Professors Michael Gage and Nachman on trying to determine the presence of an "obstacle" in an object, without prior knowledge of the background medium. This has lead to an interesting basis and dual basis for the space of square-integrable functions on the boundary of an arbitrary domain, generalizing the notion of a Fourier series expansion of a function on the unit circle.

Professors Gage, Nachman and myself were awarded a three year NSF grant to work on inverse problems in geometry and partial differential equations. As well as the works mentioned above, we have a program of research including problems in boundary rigidity of Riemannian metrics, a time-dependent hyperbolic boundary value problem for a fixed source on the boundary, determination of cracks within an inhomogeneous medium, and infinite order boundary determination of Riemannian metrics.

Dr. Jenn Nan Wang and I have proposed investigating the question of determination of chirality in an electromagnetic body from *bi-static* scattering information.

2. Outline of Previous Work

In [11], Sylvester and Uhlmann proved that the conductivity of a body can be uniquely identified from information obtained only from the boundary. In [4], Lee and Uhlmann showed that the full Taylor series of the conductivity can be determined at the boundary of a body from the same information. If a time dependence is introduced to the electromagnetic fields, the equations governing these fields change from a single second order elliptic partial differential equation to the full Maxwell's equations. In [10] Somersalo et al. presented a boundary map for time-harmonic fields at a fixed frequency and raised the question of whether the parameters describing the electromagnetic properties of the body could be determined from knowledge of this boundary map. In [8] it was shown that the parameters are recoverable provided they are known in a small neighborhood of the boundary of the body.

In all these treatments, the constituent equations, describing the dependence of the electric displacement and the magnetic induction on the electromagnetic fields, do not take into account the *chirality* of the body. Chirality is an asymmetry in the molecular structure. Presence of chirality results in the rotation of the fields. Such observations are used in physical chemistry to characterize molecular structures.

The results of my dissertation pertain to including chirality and proving that knowledge of a boundary map uniquely determines the chirality β as well as the conductivity γ, and the electric permittivity σ assuming the magnetic permeability μ to be known. All parameters are assumed to be smooth and *isotropic* (that is, scalar). Maxwell's equations for a chiral body are presented in the following boundary value problem.

THEOREM 1. *Let* $F \in TH^{\frac{1}{2}}_{\text{Div}}(\partial\Omega) = \{F \in H^{\frac{1}{2}}(\partial\Omega)^3 \mid \nu \cdot F = 0,\ DivF \in H^{\frac{1}{2}}(\partial\Omega)^3\}$ *There is a discrete set* D *containing no limit points in* $(0, \omega_0)$ *such that for all* $\omega \in (0, \omega_0)\backslash D$ *there exist unique* $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$ *solving the following boundary value problem:*

$$\begin{cases} \nabla \wedge E = i\omega(\mu H - \beta E) \\ \nabla \wedge H = -i\omega(\varepsilon E + \beta H) \\ \nu \wedge E|_{\partial\Omega} = F. \end{cases} \tag{1}$$

Here ν *is the outer unit normal vector on* $\partial\Omega$.

We may thus define the *boundary admittance map* $\Pi : TH^{\frac{1}{2}}_{\text{Div}}(\partial\Omega) \to TH^{\frac{1}{2}}_{\text{Div}}(\partial\Omega)$ as follows. Given $F \in TH^{\frac{1}{2}}_{\text{Div}}(\partial\Omega)$ let (E, H) solve (1) and define

$$\Pi F = \Pi(\nu \wedge E|_{\partial\Omega}) = \nu \wedge H|_{\partial\Omega}.$$

2.1. **Boundary Determination of Electromagnetic Parameters.** [5], [3]

The results in inverse boundary value problems tend to require knowledge of parameters on the boundary of the body, as assumed in [8]. In [5] I proved that from Π we can reconstruct all parameters together with their first normal derivatives at the boundary. We rewrite Maxwell's equations as a system of second order equations $\mathcal{M}(x, D)$ with principal part the Laplacian, and isolating the normal direction, consider this as a *quadratic* equation in normal differentiation. This quadratic is then factorized in terms of a pseudodifferential operator B which is of first order in the tangential variables x', and depends smoothly on the normal variable x_3. This factorization is done modulo smoothing:

$$\mathcal{M}(x, D_{x'}) = (D_{x_3} - iP(x) - iB(x, D_{x'}))(D_{x_3} + iB(x, D_{x'}))$$

modulo smoothing, where P is a matrix multiplier, and $D = -i\partial_x$. We then show that the boundary map Π can be expressed as a pseudodifferential operator in terms of the operator B. Computing the two terms of highest order homogeneity in an asymptotic expansion for the symbol of Π, we find that these determine all four parameters and their normal derivatives, at the boundary. This can be shown to be sufficient to remove the boundary assumption in [8].

The main difficulties that arise are in the complexity of the computations in this case of a system of operators, and the problem of handling a general non-flat boundary. The latter is overcome for the first two terms in the symbol expansion by choosing local coordinates which are flat to first order in the tangential directions.

In [3] Mark Joshi and I proved that Π in fact determines all four parameters *to infinite order* at the boundary. The breakthrough here was the introduction of a new class of pseudodifferential operators which can be used to isolate buried information in a complex asymptotic calculation as the *principal* symbol of a new operator. Computation of this principal symbol is significantly simpler, and can be shown to iteratively determine all normal derivatives of the unknown parameters at the boundary. We were able to handle the geometry of an arbitrary (smooth) non-flat boundary for the determination of every derivative.

These papers are available online from my personal home-page, following the "research" link: `www.math.rochester.edu/u/mcdowall`.

2.2. **An Electromagnetic Inverse Problem in Chiral Media.** [6]

The main theorem is the following:

THEOREM 2. *Let* $(\Omega; \varepsilon_1, \mu, \beta_1)$ *and* $(\Omega; \varepsilon_2, \mu, \beta_2)$ *be two electromagnetic bodies with smooth electromagnetic parameters, and with the same smooth boundary* $\partial\Omega$*. Suppose that* $\Pi_1 = \Pi_2$*; that is, if* $F \in TH^{\frac{1}{2}}_{Div}(\partial\Omega)$ *and* (E_j, H_j) *solve* (1) *with parameters* $(\varepsilon_j, \mu, \beta_j)$ *for* $j = 1, 2$*, then*

$$\Pi_1 F = \nu \wedge H_1|_{\partial\Omega} = \nu \wedge H_2|_{\partial\Omega} = \Pi_2 F.$$

Then throughout Ω*,* $\varepsilon_1 = \varepsilon_2$*, and* $\beta_1 = \beta_2$*.*

We show that if (E_j, H_j) solve Maxwell's equations for parameters $(\varepsilon_j, \mu_j, \beta_j)$, $j = 1, 2$, and if $\Pi_1 = \Pi_2$, then

$$\int_\Omega \big((\beta_1 - \beta_2)(H_1 \cdot E_2 + H_2 \cdot E_1) + (\varepsilon_1 - \varepsilon_2)E_1 \cdot E_2 + (\mu_2 - \mu_1)H_1 \cdot H_2\big) = 0 \tag{2}$$

We must show that there are sufficiently many solutions of suitable a form to conclude that each component of the integrand in (2) is zero. The technique is to use *complex geometrical optics* in the manner of [11] and many subsequent papers; that is we construct exponentially growing solutions depending on a complex parameter ρ and examine the asymptotics as the size of ρ gets large. Rather than construct solutions to Maxwell's equations directly, we follow the idea of Ola and Somersalo in [9] and introduce a new 8×8 system

$$(P(\nabla) + V)(P(\nabla) + V')Y = (\Delta + N + Q)Y = 0$$

where $P(\nabla)$ and N are first order differential operators, and V, V' and Q are matrix multipliers. This is done in such a way that if Y is a solution to this system, and $X = (P(\nabla) + V')Y$ is such that the first and last components of X are zero, then the vector fields $(X_2, X_3, X_4)'$ and $(X_5, X_6, X_7)'$ solve Maxwell's equations.

We construct exponentially growing solutions to $(\Delta + N + Q)Y_\rho = 0$ of the form $Y_\rho = e^{x \cdot \rho}(y_{0,\rho} + \psi_\rho)$ with $\rho \in \mathbb{C}^3$ satisfying $\rho \cdot \rho = \omega^2 \varepsilon_0 \mu_0$, with $y_{0,\rho}$ an 8-vector which is constant in x and chosen to depend on ρ in a convenient way, and ψ_ρ constructed so that $\psi_\rho \to 0$ in some sense as $|\rho| \to \infty$. In [9] where chirality was not taken into account ($\beta = 0$) the system above included no first order term N, and so the authors were able to use the methods of [11] to construct exponentially growing solutions to a Schrödinger equation. When $\beta \neq 0$, such a reduction does not seem possible, and so here we must construct solutions to a first order perturbation of the Laplacian.

The final ingredient is to set $X_\rho = (P(\nabla) + V')e^{x \cdot \rho}(y_{0,\rho} + \psi_\rho)$ and show that we can choose $y_{0,\rho}$ in such a way that X_ρ yields solutions to Maxwell's equations, and to use these solutions in (2) to prove the claim of theorem 2.

Construction of the solutions Y_ρ requires an involved microanalytic approach, first introduced in [7] in the context of an inverse problem in elasticity. We consider the operator $\Delta + N + Q$ conjugated by $e^{x \cdot \rho}$, denoted $\Delta_\rho + N_\rho^+$, and construct "intertwining operators" A_ρ, B_ρ, and C_ρ such that

$$(\Delta_\rho + N_\rho^+)A_\rho = B_\rho(\Delta_\rho + C_\rho).$$

The right hand side can be inverted, and A_ρ is constructed to be invertible for large ρ, so we have a means to construct the desired solutions. Construction of A_ρ involves dividing phase-space into the region where $\Delta_\rho + N_\rho^+$ is elliptic, and its complement, and then constructing A_ρ in each part separately. We then define B_ρ and C_ρ in terms of A_ρ.

The remaining difficulty is in isolating each term in the integrand of (2). This is achieved by careful choice of ρ and $y_{0,\rho}$, and computing the highest order terms in the dot products of the fields in this integrand. The fields are of order one in $|\rho|$ and so we might expect order two terms in the products; this fails to be the case, and we are left needing to compute the terms of order one in the products. These computations are very complex and it is non-trivial to extract the claim of the theorem. It is in this computation that the necessity to assume that the magnetic permeabilities are equal arises.

This paper is available online from my personal home-page, following the "research" link: `www.math.rochester.edu/u/mcdowall`.

3. Ultrasound Imaging

The major research project with which I am currently involved is in collaboration with Professor Adrian Nachman, and further with Professor Robert Waag of the Ultrasound Research Laboratory, department of electrical and computer engineering, and his colleagues. Waag's laboratory possesses a state of the art ultrasound instrument capable of obtaining outstandingly high precision measurements in a large frequency range (typically 1 to 5 MHz). Nachman's involvement with Waag goes back many years, and it has been a unique and exciting opportunity to join their team. During the summer of 1999, I was funded for one summer month by one of the ultrasound research lab.'s NIH grants. This collaboration has provided an excellent opportunity to learn a great deal about the engineering approach, about their skills and intuition, about numerical implementation, and about the extensive opportunities available for constructive collaboration between our disciplines.

The ultrasound instrument is a 150mm diameter ring of 2048 transducers. With this apparatus, arbitrary prescribed wave forms can be transmitted, received waves recorded, and done so with any transmit and receive aperture able to be synthesized. Inherent in such an instrument is the problem of the ring itself acting as a boundary reflector resulting in waves bouncing back and forth within the ring, requiring a very complicated modeling boundary value problem. This problem is removed by the inclusion of a lens which serves two very important functions. Firstly it is absorbing, and so any waves which return to the ring do not get reflected back, and secondly it focuses the waves into a plane, despite their being generated in a cylinder. Furthermore, the material properties of the lens are such that it is impedance matched to the background medium (water) so that there is no jump discontinuity in the medium at the interface. Because of the absorption of waves, the physical dynamics may be modeled as a *scattering* problem, rather than a boundary value problem. It is possible to extrapolate measurements made at the transducer ring to *far-field* measurements, as if the measurements were made at a great distance from the ring.

The mathematical, and practical, problem in question is that of reconstructing an image from knowledge of the scattering amplitude in a certain frequency range. The partial differential equation modeling this system is the wave equation, with the wave initially prescribed *at infinity*, and the scattered wave observed *at infinity*. This is a significant departure from the elliptic boundary value problem considered in my dissertation, and so I have had the opportunity and necessity to learn and research in another area of inverse problems. This refocusing has been stimulating and challenging, and the opportunity to do so is one which has always attracted me to research.

The ultrasound lab has very high quality *phantom* test objects with background medium having different sound speed from water, and with further inhomogeneities within. These are generally 6 to 48 mm in diameter. Currently, the principal method for reconstructing an image from data is using the Born approximation, the result of linearizing the non-linear problem. For the objects of interest here, the Born approximation performs poorly; it is insufficient for recovering the inner structure of our objects.

We are currently investigating a stable numerical procedure for non-linear inversion which goes beyond the Born approximation. Waag's lab has developed an accurate and fast forward solver which can calculate synthetic scattering data for inhomogeneous media. This solver cleverly *absorbs* waves at the boundary of the region of computation, thus removing reflections from this artificially imposed boundary. We plan to use this forward solver in combination with recent and promising ideas on stabilization of *layer stripping* algorithms. The forward solver uses a technique of marching simultaneously in time and in Fourier space, and we are investigating the use of a similar marching technique in the inverse problem. This is an ambitious and promising research program with far-reaching potential. The development of a working numerical method for acoustic imaging is a highly sought after goal.

4. Determination of a Riemannian Metric from Boundary Values of Harmonic Forms

Let M be a smooth Riemannian manifold with smooth boundary ∂M, and let $g = (g_{ij})$ be the Riemannian metric tensor on M. We denote by $\Omega^p(M)$ the space of smooth p-forms over M, and by $\Omega^p(\partial M)$, the space of smooth p-forms on ∂M. The metric g induces a volume form $\sigma \in \Omega^n(M)$, and hence a Hodge-star operator $* : \Omega^p(M) \to \Omega^{n-p}(M)$ defined by the requirement that for any $u \in \Omega^p(M)$, $*u \wedge u = \langle u, u \rangle_g \, \sigma$. If d is the exterior derivative, and δ the formal adjoint with respect to the induced L^2-product

$$(u, v)_g = \int_M u \wedge *v,$$

then the Laplace-Beltrami operator on $\Omega^p(M)$ with respect to the metric g is

$$\Delta_g = d\delta + \delta d : \Omega^p(M) \to \Omega^p(M).$$

A form $u \in \Omega^p(M)$ is called a harmonic form if $\Delta_g u = 0$.

For 0-forms one has the so called *Dirichlet to Neumann* map Λ_g which maps a function f on ∂M to the normal derivative of u at the boundary, where u solves the boundary value problem

$$\begin{aligned} \Delta_g u &= 0 \qquad \text{in } M \\ u|_{\partial M} &= f. \end{aligned}$$

It is known that knowledge of Λ_g does not uniquely determine the metric g. Indeed, if Φ is any diffeomorphism of M which is the identity on the boundary, and g' is the pull-back of

g under this diffeomorphism, then $\Lambda_g = \Lambda'_g$. It is an open problem as to whether this is the only obstruction to uniqueness.

One may ask the analogous question for p-forms. In [1] the following boundary value problem is shown to be well posed:

THEOREM 3. *Given $\varphi \in \Omega^p(\partial M)$ and $\psi \in \Omega^{n-p}(\partial M)$ there exists a unique p-form $u \in \Omega^p(M)$ solving the following boundary value problem:*

$$\Delta_g u = 0 \quad \text{in } M \tag{3}$$

$$\iota^* u = \varphi \tag{4}$$

$$\iota^*(*u) = \psi \tag{5}$$

We can now define the following boundary map

$$\Lambda_g : \Omega^p(\partial M) \times \Omega^{n-p}(\partial M) \to \Omega^{n-p-1}(\partial M) \times \Omega^{p-1}(\partial M).$$

Given $(\varphi, \psi) \in \Omega^p(\partial M) \times \Omega^{n-p}(\partial M)$ let $u \in \Omega^p(M)$ solve (3)-(5), and define

$$\Lambda_g(\varphi, \psi) = (\iota^*(*du), \iota^*(*d * u)) = (\Lambda_g^1 \varphi, \Lambda_g^2 \psi).$$

One may now ask whether or not knowledge of Λ_g for p-forms (p fixed) uniquely determines the metric g. In full generality this is arguably a more difficult question than the open question for Λ_g on functions. I am currently studying the following restricted conjecture:

CONJECTURE 4. *Suppose that $M \subset \mathbb{R}^n$ is a domain with smooth boundary ∂M. Let (M, g) and $(M, \tilde{g})$ be M equipped with the metrics g and $\tilde{g}$ respectively, and assume that g and $\tilde{g}$ are conformal to the Euclidean metric e. If the boundary maps Λ_g and $\Lambda_{\tilde{g}}$ are equal on $\Omega^p(\partial M) \times \Omega^{n-p}(\partial M)$ then $g = \tilde{g}$ throughout M.*

Mark Joshi and William Lionheart [2] applied the boundary determination techniques of Joshi-McDowall [3] to the case of determining the metric at the boundary from knowledge of Λ_g and proved that indeed the metric is uniquely determined to infinite order at the boundary.

The key starting point for proving the above conjecture for determination of g in the interior of M is the following integral identity: if $u,\ \tilde{u} \in \Omega^p(M)$ solve $\Delta_g u = 0$ and $\Delta_{\tilde{g}} \tilde{u} = 0$, and if $\Lambda_g = \Lambda_{\tilde{g}}$, then

$$\int_M \Big((-1)^{np} d * u \wedge \tilde{*} d \tilde{*} \tilde{u} - d(\lambda^{-p+n/2} u) \wedge \tilde{*} d\tilde{u} \\ - (-1)^{np} d \tilde{*} \tilde{u} \wedge * d * u + d(\lambda^{p-n/2} \tilde{u} \wedge * du \Big) = 0. \tag{6}$$

Here, $\tilde{*}$ is the Hodge-star operator for the metric $\tilde{g}$.

I am able to construct exponentially growing solutions to $\Delta_g u = 0$ using techniques of microlocal analysis. It remains to show that there are sufficiently many such solutions to conclude from (6) that $g = \tilde{g}$.

I expect this to open a number of avenues of future research along these lines. It is possible that the boundary operator Λ_g is in some sense *richer* when known on p-forms and may yield access to identifiability results for more general manifolds, and more general metrics.

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